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THE BOOLEAN ALGEBRA OF THE THEORY OF LINEAR ORDERS†

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ABSTRACT

We characterize the isomorphism type of the Boolean algebra of sentences of the theory of linear orders. It is isomorphic to the sentence algebras of the theory of equivalence relations, the theory of permutations and the theory of well-orderings.

Introduction

For any linear order type τ with first element, let $\mathfrak{A}(\tau)$ be the Boolean set algebra generated by the left-closed right-open (including $[x,\infty)$) intervals of some linear order of type τ. Let η and ω be the order types of rationals and nonnegative integers respectively. We show that the Boolean sentence algebra of the elementary theory of linear orders, i.e., the Boolean algebra of elementary classes of linear orders or, equivalently, the algebra of equivalence classes of sentences of the elementary theory of linear orders, is isomorphic to $\mathfrak{A}(\omega^\omega(1+\eta))$. By earlier work of Hanf [3], Simons [9], and Myers [7], this is also the Boolean sentence algebra of the theory of equivalence relations, the theory of permutations, and the theory of well-orderings.

The proof uses a dual of Hanf's structure diagrams [3] to classify the Boolean algebra and an equivalence relation of Shelah's [1] to classify the linear orders.

We prefer "Boolean sentence algebra" to "Lindenbaum algebra": see [5] for the historical justification.

Rank diagrams

Rank diagrams are dual to Hanf's structure diagrams [3]. If a totally transcendental (empty perfect hull) Boolean space which is pseudo-

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indecomposable with respect to disjoint union ($x = y \dot{\cup} z \Rightarrow x \simeq y$ or $x \simeq z$) has Cantor–Bendixson rank α then α, regarded as a partial order, is a rank diagram for the space. While the Cantor–Bendixson rank is a complete invariant only for pseudo-indecomposable totally transcendental spaces, rank diagrams are complete invariants for all primitive (a space is primitive iff it is a pseudo-indecomposable space whose clopen sets are disjoint unions of pseudo-indecomposable clopen sets, see Hanf [3] for details) Boolean spaces with a countable basis.

Let $\mathcal{R} = \langle R, < \rangle$ be a transitive antisymmetric relation structure with a largest element. We will call it a partial order even though it is not necessarily reflexive or irreflexive and we will call its elements ranks. Let $\leqq$ and $\lneqq$ be the associated reflexive and irreflexive relations. A rank in $\mathcal{R}$ is separated iff it is not the infimum of a chain of ranks strictly above it. Ranks p and q are *incomparable* iff $p \not< q$ and $q \not< p$ and *strictly incomparable* if in addition $p \neq q$. A sequence q_i is *strictly increasing* iff $q_i \lneqq q_{i+1}$; likewise for *strictly decreasing*.

Definition 1. For any Boolean space $\mathcal{X}$ with a countable basis and any countable partial order $\mathcal{R}$ with largest rank, $\mathcal{R}$ is a *rank diagram* for $\mathcal{X}$ iff there is an onto function $f: \mathcal{X} \to \mathbf{R}$ such that for all $p, q \in \mathbf{R}$, $x, x_0, x_1, \cdots, \in \mathcal{X}$:

(1) if $p > q$ and $f(x) = p$, then ($\exists$ an ω-sequence x_i) $x = \lim x_i$, $x_i \neq x$, and $f(x_i) = q$;

(2) if $x = \lim x_i$, $x_i \neq x$, and $f(x_i) = q$, then $f(x) > q$;

(3) if q is separated, $f(\lim x_i) \leqq q$, and the $f(x_i)$'s are pairwise strictly incomparable or strictly decreasing, then

$$f(x_i) \leqq q \text{ for all but finitely many } i;$$

(4) if $f(x)$ is not separated, then ($\exists$ an ω-sequence x_i) $x = \lim x_i$ and $f(x_i) \gneqq f(x)$;

(5) if q is the largest rank and $q \not< q$, then $(\exists!\, x \in \mathcal{X})\ f(x) = q$.

If f is as in the definition, $\mathcal{R}$ is said to *rank* $\mathcal{X}$ *via* f.

Example. Let $\mathcal{X}$ be a primitive Boolean space with a countable basis and let R be the set of orbits $\bar{x}$ of points x in $\mathcal{X}$ acted on by homeomorphisms. Let R be ordered by $\bar{x} < \bar{y}$ iff every $y \in \bar{y}$ is a limit of distinct points from $\bar{x}$. Then $\langle R, < \rangle$ is a rank diagram for $\mathcal{X}$ via $f(x) = \bar{x}$.

For the rest of this section suppose $\mathcal{X}$ is a Boolean space with a countable basis ranked via f by a countable partial order $\mathcal{R}$. For $x \in \mathcal{X}$, $f(x)$ is called the *rank* of x. For $A \subseteq \mathcal{X}$ and $x \in A$: x is a *max point* of A and $f(x)$ is a *max rank* of A iff there is no $y \in A$ such that $f(x) \lneqq f(y)$, and x is a *strict max point* iff $f(y) < f(x)$ for all $y \in A$ not equal to x.

LEMMA 2. (a) *A max point of a clopen set has separated rank.*

(b) *A point of separated rank is a strict max point of some clopen set and, if x is of separated rank and $f(x') < f(x)$, then x is a strict max point of some clopen set containing x'.*

PROOF. (a) By (4) a point in a clopen set of nonseparated rank is not maximal.

(b) Suppose x has a separated rank and $f(x') < f(x)$. Let $Y = \{y \in \mathscr{X} \mid y \neq x$ and $f(y) \not< f(x)\}$. We show that $x \notin$ the closure of Y. Suppose $x = \lim y_i$ for some $y_i \in Y$. By Ramsey's Theorem [8] and possibly taking a subsequence we may assume that the $f(y_i)$'s are (i) equal, (ii) strictly increasing, (iii) strictly incomparable, or (iv) strictly decreasing. In case (i) and (ii) x is a limit of distinct points of rank $f(y_0)$ (case (ii) needs condition (1) of the definition which implies that each y_i and hence x is a limit of rank-$f(y_0)$ points). Hence $f(y_0) < f(x)$ by (2) contradicting $y_0 \in Y$. In case (iii) and (iv) condition (3) requires that $f(y_i) < f(x)$ for almost all i contradicting $y_i \in Y$. The same argument shows $x' \notin$ the closure of Y since $f(y_0)$ or $f(y_i) < f(x')$ implies $f(y_0)$ or $f(y_i) < f(x)$. Hence x and x' are separated from Y by some clopen set A. But then x is a strict max point of A.

LEMMA 3. *The rank of any point in a clopen set is $\leqq$ a max rank of the set.*

PROOF. By Zorn's Lemma and $\mathscr{R}$'s countability, it suffices to show that every ω-chain of ranks of points of the clopen set has an upper bound. Suppose $f(x_0) \lneqq f(x_1) \lneqq f(x_2) \cdots$. By compactness and possibly taking a subsequence we may suppose $\lim x_i$ exists. By (1) of the definition, each $x_j, j > i$ is a limit of points of rank $f(x_i)$. Hence so is $\lim x_i$ and hence, by (2), $f(x_i) < f(\lim x_i)$.

Let $\mathfrak{B}$ be the Boolean algebra of clopen subsets of $\mathscr{X}$. A subset C of $\mathfrak{B}$ *disjointly generates* $\mathfrak{B}$ iff every nonempty element of $\mathfrak{B}$ is a disjoint union (let $a \dot{\cup} b$ denote disjoint union) of elements of C. Let M be the set of clopen subsets of $\mathscr{X}$ which have a strict max point.

LEMMA 4. *M is a set of disjoint generators for $\mathfrak{B}$.*

PROOF. Let A be any clopen subset of $\mathscr{X}$. Suppose $x \in A$. By Lemma 3, $f(x) \leqq f(y)$ for some max point $y \in A$. By Lemma 2, $f(y)$ is separated and y is a strict max point of some clopen set B containing x. Hence every point of A is in a clopen subset, $B \cap A$ for example, of A with a strict max point. By compactness $A = B_1 \cup \cdots \cup B_n$ for some clopen sets B_i, with strict max points $x_1, \cdots, x_n$ respectively. If $f(x_j) < f(x_i)$ then x_i is a strict max point of $B_i \cup B_j$ and hence we may omit B_j and replace B_i by $B_j \cup B_i$. Repeating this process we obtain $A = B_1 \cup \cdots \cup B_n$ where $f(x_i)$ and $f(x_j)$ are incomparable for $i \neq j$. Let $A_i \subseteq B_i$ be clopen sets such that $A = A_1 \dot{\cup} \cdots \dot{\cup} A_n$. Since $x_i \notin B_j$ for $j \neq i$, $x_i \in A_i$ and hence the A_i's have strict max points and are in M.

Let $\mathcal{S} = \langle\{q \in \mathcal{R} \mid q \text{ is separated}\}, <\rangle$. Define $F: M \to \mathcal{S}$ by $F(A) =$ the unique max rank of A.

LEMMA 5. *$\mathcal{S}$ structures* (see Hanf [3] or Myers [7]) $\mathfrak{B}$ *via F, i.e., F is onto, F's domain disjointly generates* $\mathfrak{B}$ *and contains* 1 *but not* 0, *and for all* $A, B, C, B_1, \cdots, B_n \in M$ *and all* $p \in \mathcal{S}$:

(i) $B \subseteq A$ *implies* $F(B) \lesseqgtr F(A)$,

(ii) $B \dot{\cup} C \subseteq A$ *and* $F(B) = F(C) = F(A)$ *implies* $F(A) < F(A)$,

(iii) $p < F(A)$ *implies* $(\exists D, E \in M)$ $D \dot{\cup} E \subseteq A$, $F(D) = p$, *and* $F(E) = F(A)$,

(iv) $B_1 \dot{\cup} \cdots \dot{\cup} B_n = A$ *implies* $F(B_1) = F(A)$ *or ... or* $F(B_n) = F(A)$.

PROOF. F is onto by Lemma 2. M disjointly generates $\mathfrak{B}$ by Lemma 4. $\mathcal{X} \in M$ by the existence of a largest and thus unique rank in $\mathcal{R}$ and hence in $\mathcal{S}$ and by (5). $0 \notin M$ by definition of M. Of the four conditions, (i) and (iv) are clear. For (ii), if $F(A) \not< F(A)$ then A has at most one point of rank $F(A)$ and so $F(B) \neq F(A)$ or $F(C) \neq F(A)$. To show (iii) suppose $p < F(A)$. Let $y \in A$ have rank $F(A)$. By (1), there is a point $x \neq y$ of rank p in A. By Lemma 2 and the separateness of p, x is a max point of some clopen subset D of A. We may assume $y \notin D$. Let $E = A \sim D$. Then $D \dot{\cup} E \subseteq A$, $F(D) = p$ and, since $y \in E$, $F(E) = F(A)$.

THEOREM 6. *If $\mathcal{X}$ and $\mathcal{Y}$ are Boolean spaces with countable bases and with a common rank diagram, then $\mathcal{X} \simeq \mathcal{Y}$.*

PROOF. By Lemma 5, if $\mathcal{X}$ and $\mathcal{Y}$ have a common rank diagram, their dual algebras have a common structure diagram and, by a theorem of Hanf [3], are isomorphic.

$\mathfrak{A}(\omega^\omega(1+\eta))$. Of all Boolean algebras, interval algebras are among the easiest to visualize. Let τ be a linear order with a first element and let $\mathfrak{A}(\tau)$ be the Boolean algebra of left-closed, right-open intervals as described in the introduction. A cut is a splitting of τ into two segments (convex sets) such that the lower half is nonempty. A cut is in an interval $[a, b)$ (b may be ∞) if a is in the lower half and b is in the upper half (∞ is considered to be in the upper half of every cut). Thus each interval of τ determines an interval of cuts. Let $\mathscr{X}(\tau)$ be the Boolean space whose universe is the set of cuts in τ and whose topology is generated by the above intervals of cuts. The obvious 1-1 correspondence between cuts in τ and maximal ideals of $\mathfrak{A}(\tau)$ makes $\mathscr{X}(\tau)$ isomorphic to the dual space of $\mathfrak{A}(\tau)$.

THEOREM 7. *$\mathscr{X}(\omega^\omega(1+\eta))$ is ranked by $\langle \omega+1, <\rangle = \langle\{0,1,2,\cdots,\omega\}, < \bigcup \{\langle\omega,\omega\rangle\}\rangle$ where $<$ is the usual ordinal order.*

PROOF. Let f: $\mathscr{X}(\omega^\omega(1+\eta)) \rightarrow \{0,1,\cdots,\omega\}$ be the function which assigns rank 0 to isolated cuts, rank 1 to limits of rank 0 cuts which are not limits of nonzero rank cuts, $\cdots$, rank $n+1$ to limits of rank n cuts which are not limits of cuts whose rank is not $\leqq n, \cdots$, and rank ω to all other cuts. Regarding $\omega^\omega(1+\eta)$ as a dense union of segments of type ω^ω, finite ranks are assigned to cuts occurring within ω^ω-type segments and ω is assigned to cuts occurring between ω^ω-type segments. We show that $\langle\omega+1, <\rangle$ ranks $\mathscr{X}(\omega^\omega(1+\eta))$ via f. Since all ranks are separated, separated ranks are dense. Parts (1) and (2) of Definition 1 follow from the way finite ranks were assigned and the fact that each cut of rank ω is a limit of cuts of any given rank. Parts (3) and (4) are trivial since there are no strictly incomparable ranks, no infinite strictly decreasing sequences of ranks, and no nonseparated rank. Part (5) is trivial since $\omega < \omega$.

Linear orders, games, and definability

Let **Lin** = the Boolean space of linear orders (modulo elementary equivalence which is denoted by $\equiv_\omega$) under the elementary topology (closed sets = EC_Δ classes). This is the dual space of the Boolean sentence algebra of the theory of linear orders.

In a given linear order α, a *segment* is a convex subset and a *splitting* is an equivalence relation all of whose equivalence classes are segments. A splitting $\sim$ is *finite* iff it has finitely many segments and it is *definable* iff for some formula $\varphi(x, y)$ of the first-order language of linear orders $x \sim y$ iff $\alpha \models \varphi(x, y)$. Given a splitting $\sim$ of α, $\alpha/\sim$ is the quotient structure and we will say α is infinite, has no last point, is dense, $\cdots$, *mod* $\sim$ iff $\alpha/\sim$ is infinite, has no last point, is dense, $\cdots$. A segment β of α is $\sim$*-closed* iff $x \in \beta$ and $y \sim x$ implies $y \in \beta$. The $\sim$*-closure* of a segment β of α is the smallest $\sim$-closed segment of α including β.

A *splitting assignment* is a function on **Lin** which assigns to an order α a splitting $\sim_\alpha$ (sometimes abbreviated to $\sim$) of α. An assignment $\sim$ is definable iff for some formula $\varphi(x, y)$, $x \sim_\alpha y$ iff $\alpha \models \varphi(x, y)$ for all linear orders α.

For any orders $\tau_1, \cdots, \tau_n$, let $\eta(\tau_1, \cdots, \tau_n) = \Sigma_{r \in Q}\sigma_r$ where $\sigma_r \in \{\tau_1, \cdots, \tau_n\}$ for each $r \in Q$ and $\{r \mid \sigma_r \cong \tau_i\}$ is dense in Q for each $i = 1, \cdots, n$. More generally, any $\Sigma_{r \in Q'}\sigma_r$ where Q' is dense and without endpoints and $\{r \mid \sigma_r \equiv_\omega \tau_i\}$ is dense for each i is called a shuffle product of $\tau_1, \cdots, \tau_n$.

For any φ, let quant(φ) be the quantifier depth of φ. An *m-sentence* is a sentence with quantifier depth m.

An Ehrenfeucht–Fraïssé m-game between two linear orders α and β is a game of m rounds between two players, player I and player II, such that in each round player I selects a point from α or β and player II selects a matching point from the opposite order. Player II wins if after m rounds the resulting matching is a partial isomorphism; otherwise player I wins. We say α and β are m-equivalent and write $\alpha \equiv_m \beta$ iff player II has a winning strategy for m-games between α and β. For each m, $\equiv_m$ is an equivalence relation and $\alpha \equiv_m \beta$ iff α and β satisfy the same m-sentences. There are only finitely many $\equiv_m$-equivalence classes. These equivalence classes will be called m-types and an ω-equivalence, i.e., elementary equivalence, class will be called an ω-type.

An order α is isolated in **Lin** iff its theory is finitely axiomatizable. α is *m-isolated* iff it is isolated by (the clopen set of models satisfying) an m-sentence.

LEMMA 8. *For any linear orders I, J, α_i for $i \in I$, and β_j for $j \in J$: If $I \equiv_m J$ via some strategy S and $\alpha_i \equiv_m \beta_j$ whenever (i, j) is a matching pair picked by S, then $\Sigma_{i \in I}\alpha_i \equiv_m \Sigma_{j \in J}\beta_j$.*

PROOF. Given a point $\alpha \in \alpha_i$, use strategy S to pick a matching point $j \in J$ and use a selected strategy for $\alpha_i \equiv_m \beta_j$ to pick a matching $y \in \beta_j$.

COROLLARY 9. *If $I \equiv_m J$ then $\alpha \cdot I \equiv_m \alpha \cdot J$ and if $\alpha_i \equiv_m \beta_i$ then $\Sigma_{i \in I}\alpha_i \equiv_m \Sigma_{i \in I}\beta_i$.*

LEMMA 10. *If β is a segment of α, then the restriction to β of a relation definable in α is definable in β.*

PROOF. Suppose R is the restriction to β of a relation defined by φ in α. Suppose R is not definable in β. Then, by Beth's theorem, there are structures (γ, R') and (γ, R'') elementarily equivalent to (β, R) such that $R \neq R''$. Let $\alpha = \alpha_1 + \beta + \alpha_2$. With a strategy similar to that used in Lemma 8, we can show for any m,

$$(\alpha_1 + \beta + \alpha_2, \beta, R) \equiv_m (\alpha_1 + \beta' + \alpha_2, \beta', R') \equiv_m (\alpha_1 + \beta' + \alpha_2, \beta', R'')$$

where β, for example, is being used to stand for the unary relation it determines. Thus the three structures are elementarily equivalent and, since $(\forall \bar{x} \in \beta)$ $R\bar{x} \Leftrightarrow \varphi(\bar{x}))$ holds in the first, it holds in all three. But then $R' = R''$, a contradiction.

LEMMA 11. *If β is a segment of a definable splitting of α, then any relation definable in β is the restriction to β of a relation definable in α.*

PROOF. Suppose φ defines a splitting of α of which β is a segment and suppose $\psi(x, y, \cdots)$ defines a relation in β. Then the desired relation on α is defined by $\psi^x(x, y, \cdots)$ where ψ^x is the relativization of ψ to x's ψ equivalence class.

LEMMA 12. *For any m, any splitting assignment $\sim$ defined by an m-formula, and any splitting $\sim^*$ of a linear order β: If for some linear order α there is a bijective order-preserving correspondence between the segments of $\sim_\alpha$ and those of $\sim^*$ such that corresponding segments are $(m+2)$-equivalent, then $\sim^* = \sim_\beta$.*

PROOF. With a winning strategy similar to that used in Lemma 8, we have $(\alpha, \sim_\alpha) \equiv_{m+2} (\beta, \sim^*)$. Let u and v be any two points of β. In his first two moves let player I choose u and v and let x and y be the matching points in α picked according to the above strategy. Then, since there are still m moves left, $(\alpha, \sim_\alpha, x, y) \equiv_m (\beta, \sim^*, u, v)$. Consequently $u \sim^* v$ iff $x \sim_\alpha y$ iff, since $\sim$ is m-definable, $u \sim_\beta v$.

Orders of rank one

An order has rank 0 iff it is isolated in **Lin** iff for some $\sigma, \alpha \models \sigma$ and if $\beta \models \sigma$ then $\beta \equiv_\omega \alpha$. An order α has rank 1 iff $\alpha \equiv_\omega \lim \alpha_i$ for some isolated α_i's with $\alpha_i \not\equiv_\omega \alpha$ and α is isolated from all nonisolated orders, i.e., for some σ, $\alpha \models \sigma$ and if $\beta \models \sigma$ then either $\beta \equiv_\omega \alpha$ or β is isolated.

The archetypal order of rank 1 is $\omega + \omega^*$ (here ω^* is the reverse of ω). More generally, if α has a definable splitting $\sim$ all of whose segments are of rank 0 and are all ω-equivalent and if $\alpha/\sim\ \equiv_\omega \omega + \omega^*$, then α has rank 1.

A *finite* splitting is one with finitely many segments. A 0-1 *splitting* is one all of whose segments have rank 0 or rank 1.

LEMMA 13. (a) *If* α *has rank* 0, *then all segments of definable splittings of* α *have rank* 0.

(b) *If* α *has a finite definable splitting all of whose segments have rank* 0, *then* α *has rank* 0.

(c) *If* α *has rank* 1, *then all segments of definable splittings of* α *have rank* 0 *except at most one which may have rank* 1.

(d) *If* α *has a finite definable* 0-1 *splitting with exactly one rank* 1 *segment, then* α *has rank* 1.

PROOF. (a) Let α be a rank 0 order with a splitting defined by φ. Pick m so that α is m-isolated and $m \geqq \text{quant}(\varphi)+2$. If β is a $\sim$-segment and not isolated, then, for some β' and some $n > m$, $\beta' \equiv_m \beta$ but $\beta' \not\equiv_n \beta$. Let α' be the result of replacing each segment of $\sim$ which is m-equivalent to β by (a segment isomorphic to) β'. Then $\alpha' \equiv_m \alpha$ but $\alpha' \not\equiv_\omega \alpha$ since "there is a φ-segment n-equivalent to β" is true in α but not in α' (Lemma 12 is needed here) and can be expressed by a first-order sentence. This contradicts the fact that α is m-isolated.

(b) Suppose φ defines a splitting of α into rank 0 segments $\beta_1, \cdots, \beta_n$. Let $\sigma_1, \cdots, \sigma_n$ be sentences which isolate $\beta_1, \cdots, \beta_n$ respectively. Then the first-order sentence equivalent to "the splitting defined by φ has exactly n segments and they satisfy $\sigma_1, \cdots, \sigma_n$ respectively" isolates α.

(c) Let α be a rank 1 order with a splitting defined by $\varphi(x, y)$. Pick m so that an m-sentence isolates α from nonisolated orders and $m \geqq \text{quant}(\varphi)+2$. Suppose β and γ are two nonisolated segments of $\sim$. Hence for some β' and some $n > m$, $\beta' \equiv_m \beta$ but $\beta' \not\equiv_n \beta$ and $\beta' \not\equiv_n \gamma$. Let α' be the result of replacing all $\sim$-segments other than γ which are m-equivalent to β by β'. By part (a), α' is not isolated since it contains a nonisolated segment γ. Also $\alpha' \equiv_m \alpha$ but $\alpha' \not\equiv_\omega \alpha$ since "there are two φ-segments, one n-equivalent to β and the other n-equivalent to γ" holds in α but not in α'. This contradicts the fact that α is isolated from nonisolated orders by an m-sentence.

(d) Suppose α is definably split by $\varphi(x, y)$ into segments $\beta_1, \cdots, \beta_n$ each of rank 0 except β_k which has rank 1. For $i \neq k$ let σ_i isolate β_i and let σ_k isolate β_k from nonisolated orders. Then, by part (b), the first-order sentence equivalent to "the splitting defined by φ has exactly n segments and they satisfy $\sigma_1, \cdots, \sigma_n$ respectively" isolates α from all nonisolated orders.

The ranking

Let $\langle \omega + 1, < \rangle = \langle \{0, 1, 2, \cdots, \omega\}, < \cup \{\langle \omega, \omega \rangle\} \rangle$ be as defined in the section on

$\mathfrak{A}(\omega^{\omega}(1+\eta))$. Let $r \subseteq \mathbf{Lin} \times (\omega+1)$ be the relation defined by: $\langle \alpha, p \rangle \in r$ iff (i) p is finite and α has a finite definable 0-1 splitting with exactly p rank 1 segments or (ii) $p = \omega$ and for all $m \in \omega$, α is m-equivalent to an order which has a definable 0-1 splitting with infinitely many rank 1 segments.

LEMMA 14 (Classification Lemma). *The domain of r is* **Lin**.

PROOF. The proof is relatively complicated. Since it is also independent of the rest of the paper, we postpone it until the end.

Let α be a linear order with a finite 0-1 splitting defined by $\varphi(x,y)$ with segments $\beta_1, \cdots, \beta_n$, p of which have rank 1. For $i \leqq n$, let σ_i be a sentence that isolates β_i if β_i has rank 0 or, if β_i has rank 1, isolates β_i from nonisolated orders. Let σ_α be a sentence which says "$\varphi(x,y)$ is a splitting with n segments which satisfy $\sigma_1, \cdots, \sigma_n$ respectively". (Note monadic second-order quantifiers ranging over segments of definable splittings are reducible to first-order quantifiers.) Then $\alpha \models \sigma_\alpha$. If $\beta \models \sigma_\alpha$ then β has a finite definable 0-1 splitting with less than or equal to p rank 1 segments, in fact, either $\beta \equiv_\omega \alpha$ or φ splits β into n rank 0 or 1 segments with less than p rank 1 segments.

LEMMA 15. *The relation r is a function from* **Lin** *to $\omega + 1$.*

PROOF. Given α, suppose $\langle \alpha, p \rangle$ and $\langle \alpha, p' \rangle$ are in r and $p < p'$. Then p is finite and so some formula φ, suppose it is an m-formula, defines a finite 0-1 splitting of α with p rank 1 segments. Let β be an order m-equivalent to α with a definable 0-1 splitting $\sim'$ with at least p' rank 1 segments (if p' is finite, let $\beta = \alpha$). Since $\beta \equiv_m \alpha$, $\beta \models \sigma_\alpha$ and hence φ defines a finite 0-1 splitting $\sim$ of β with at most p rank 1 segments. Now $\sim \cap \sim'$ definably splits each $\sim$ segment and so, by Lemma 13 (a) and (c), the number of rank 1 segments of $\sim \cap \sim'$ is less than or equal to p, the number of rank 1 segments of $\sim$. In addition, $\sim \cap \sim'$ finitely definably splits each $\sim'$ segment and so, by Lemma 13(b) and (c), the number of rank 1 segments of $\sim \cap \sim'$ is greater than or equal to p', the number of rank 1 segments of $\sim'$. This is impossible since $p < p'$.

COROLLARY 18. *The function r is well-defined with respect to elementary equivalence.*

PROOF. It is immediate from the definition of r that $\langle \alpha, p \rangle \in r$ and $\alpha \equiv_\omega \alpha'$ implies $\langle \alpha', p \rangle \in r$. The Corollary follows from this and the previous lemma.

Henceforth we shall use function notation for r and, for any order α, $r(\alpha)$ will be called the rank of α. If α has finite rank and if σ_α is the sentence defined prior to Lemma 15, then σ_α isolates α from orders of higher rank, i.e., if $\alpha \models \sigma_\alpha$ and $\beta \models \sigma_\alpha$ then $\beta \equiv_\omega \alpha$ or $r(\beta) < r(\alpha)$.

LEMMA 17. *The ranking function* r: **Lin** $\to \omega + 1$ *is onto.*

PROOF. Suppose $p \in \omega + 1$. Claim: $r((\omega + \omega^*) + \eta) \cdot p) = p$. Let $\sim$ be the splitting of $((\omega + \omega^*) + \eta) \cdot p$ defined by $x \sim y$ iff (i) all points equal to or between x and y have predecessors and successors or (ii) x and y are in a segment of type η. Then $\sim$ is a definable 0-1 splitting with exactly p rank 0 and p rank 1 segments. Hence $r(((\omega + \omega^*) + \eta) \cdot p) = p$.

The Boolean sentence algebra of linear orders

THEOREM 18 (Shelah). *Isolated orders are dense in* **Lin**.

PROOF. See Amit and Shelah [1].

THEOREM 19. **Lin** *is ranked by* $\langle \omega + 1, < \rangle = \langle \{0, 1, \cdots, \omega\}, < \cup \{\langle \omega, \omega \rangle\}\rangle$ *via* r: **Lin** $\to \omega + 1$.

PROOF. $\langle \omega + 1, < \rangle$ has a largest rank and since all ranks are separated, separated ranks are dense. By Lemma 17, r is onto. Of the five conditions of Definition 1, (3) and (4) are trivial since there are no strictly incomparable ranks, no infinite strictly decreasing sequences of ranks, and no nonseparated ranks. Condition (5) is trivial since ω, the largest rank, satisfies $\omega < \omega$.

Condition (2) (if $\alpha \equiv_\omega \lim \alpha_i$, $\alpha_i \not\equiv_\omega \alpha$, and $r(\alpha_i) = q$ then $r(\alpha) > q$). If $r(\alpha) = \omega$ then $r(\alpha) > q$ for any q and hence the condition is trivial. Suppose $r(\alpha)$ is finite. Let σ_α be a sentence as defined prior to Lemma 15 such that $\alpha \models \sigma_\alpha$ and if $\beta \models \sigma_\alpha$ then $\beta \equiv_\omega \alpha$ or $r(\beta) < r(\alpha)$. If $\alpha \equiv_\omega \lim \alpha_i$ and $\alpha_i \not\equiv_\omega \alpha$ then $\alpha_i \models \sigma_\alpha$ almost all i. Hence, since $\alpha_i \not\equiv_\omega \alpha$, $q = r(\alpha_i) < r(\alpha)$ and thus $q < r(\alpha)$.

Condition (1) (if $p > q$ and $r(\alpha) = p$ then ($\exists$ an ω-sequence α_i) $\alpha \equiv_\omega \lim \alpha_i$, $\alpha_i \not\equiv_\omega \alpha$ and $r(\alpha_i) = q$). Suppose $p > q$ and $r(\alpha) = p$. It suffices to show that for every m there is a β such that $\beta \equiv_m \alpha$, $\beta \not\equiv_\omega \alpha$, and $r(\beta) = q$. Suppose m is given.

Case for finite p. Let $\sim$ be finite definable 0-1 splitting of α with p rank 1 segments. Pick k so that $k \geqq \max(m, \text{quant}(\varphi) + 2)$ for some φ defining $\sim$. Let β be an order obtained by replacing all but q of $\sim$'s rank 1 segments by k-equivalent rank 0 segments. Then $\beta \equiv_m \alpha$ and, by Lemma 12 and the definition of r, $\beta \not\equiv_\omega \alpha$ and $r(\beta) = q$.

Case for $p = \omega$. Let β be an order m-equivalent to α for which there is a 0-1 splitting $\sim$ defined by some formula φ with infinitely many rank 1 segments. Pick k so that $k \geqq \max(m, \text{quant}(\varphi) + 2)$. Let T be a finite set of isolated

ω-types containing exactly one representative from each k-type. By making replacements (and using Lemma 12) if necessary, we may assume that the ω-type of each rank 0 segment of $\sim$ is from T.

Subcase for finite q. Let γ be the result of replacing in β all but q of $\sim$'s rank 1 segments by k-equivalent segments whose ω-types are in T. Let $\sim'$ be the splitting of γ defined by φ. Define $\sim^*$ on γ by $x \sim^* y$ iff all $\sim'$ segments containing x or y or any point between them are of a type from T. Then $\sim^*$ is definable by some formula ψ, each of the q rank 1 segments of $\sim'$ in γ is also a segment of $\sim^*$, and there are at most $q+1$ other segments. Replace, if necessary, the $q+1$ other segments by $\max(m, \mathrm{quant}(\psi)+2)$-equivalent rank 0 orders. The resulting order then has rank q and is m-equivalent but not ω-equivalent to α.

Subcase for $q = \omega$. If $\beta \not\equiv_\omega \alpha$ we are done. Otherwise replace in β one of the segments of $\sim$ by a k-equivalent rank 0 order whose ω-type is not in T. The resulting order still has rank ω and is m-equivalent but not ω-equivalent to α.

THEOREM 20 (Main Result). *The Boolean sentence algebra of the theory of linear orders is isomorphic to the algebra* $\mathfrak{A}(\omega^\omega(1+\eta))$ *generated by the right-closed left-open intervals of* $\omega^\omega(1+\eta)$.

PROOF. By Theorems 7 and 19, the dual spaces of these algebras are both ranked by $\langle \omega+1, < \rangle$. By Theorem 6 these spaces are isomorphic and hence so are the Boolean algebras.

The classification lemma: preliminaries

DEFINITION 21. For any definable splitting assignment $\sim$ and any linear order α: α has rank ω with respect to $\sim$ iff $\sim_\alpha$ is a 0-1 splitting with infinitely many rank 1 segments.

For the rest of this paper "order" shall mean "linear order" and i, j, k, m and n will be elements of ω. If an order has a definable 0-1 splitting with infinitely many rank 1 segments then it has rank ω with respect to some definable splitting assignment (if $\varphi(x, y)$ defines a splitting $\sim'$ of α then "$\varphi(x, y)$ if φ is a splitting and $x = y$ otherwise" defines a splitting assignment $\sim$ such that $\sim_\alpha = \sim'$). We say $\sim^*$ *extends* $\sim$ if $x \sim y$ implies $x \sim^* y$.

LEMMA 22. *For any definable splitting assignment* $\sim$, *any order* α, *and any* m: *If* $\sim_\alpha$ *has infinitely many rank* 1 *segments, then* α *is* m*-equivalent to an order of rank* ω *with respect to* $\sim$.

PROOF. Pick $n \geqq \max(m, \text{quant}(\varphi)+2)$ for some φ defining $\sim$. Let α' be obtained by replacing each $\sim_\alpha$ segment whose rank is not 0 or 1 by an n-equivalent order of rank 0. Then $\alpha \equiv_m \alpha'$ and, by Lemma 12, α' is of rank ω with respect to $\sim$.

DEFINITION 23. For any splitting assignment $\sim : \sim$ is *local* iff for every order α and every segment β of α, if β is $\sim_\alpha$-closed or if β has no endpoints mod $\sim_\beta$, then $\sim_\beta =$ the restriction to β of $\sim_\alpha$.

If $\sim$ is a local splitting assignment and β a $\sim_\alpha$-closed segment of α we will often omit subscripts and write, for example, $\beta/\sim$ instead of $\beta/\sim_\beta$ or $\beta/\sim_\alpha$. If $\varphi(x, y)$ defines a splitting assignment $\sim$ and all of φ's quantifiers are restricted to points between x and y, then $\sim$ is local.

LEMMA 24. *For any definable local splitting assignment $\sim$, any order α, and any m: If a segment of α is m-equivalent to an order of rank ω with respect to $\sim$, then α is m-equivalent to an order of rank ω with respect to $\sim$.*

PROOF. Let $\sim, \alpha$, and m be as in the hypothesis. Let β be a segment of α which is m-equivalent to an order γ of rank ω with respect to $\sim$. Pick $n \geqq \max(m, \text{quant}(\varphi)+2)$ for some φ defining $\sim$. Since γ has infinitely many rank 1 $\sim$-segments and $\equiv_n$ has only finitely many equivalence classes, γ has infinitely many n-equivalent rank 1 $\sim$-segments. By a compactness argument we may assume that γ has a (not necessarily consecutive) $(\omega^* + \omega)$-sequence of such segments $\cdots \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \cdots$ ordered by γ's order. Let δ be the smallest segment of γ containing the δ_i's. Then δ has infinitely many rank 1 $\sim$-segments and has no endpoints mod $\sim$. Let $\alpha = \alpha_0 + \beta + \alpha_1$, let $\gamma = \gamma_0 + \delta + \gamma_1$, and let $\alpha' = (\alpha_0 + \gamma_0) + \delta + (\gamma_1 + \alpha_1)$. Then $\alpha' \equiv_m \alpha$ and, since $\sim$ is local, every segment of $\sim_\delta$ is also a segment of $\sim_{\alpha'}$. Hence $\sim_{\alpha'}$ has infinitely many rank 1 segments and, by Lemma 22, α' and hence α is m-equivalent to an order of rank ω with respect to $\sim$.

DEFINITION 25. Let $\sim$ be a definable splitting assignment. All of the following definitions are with respect to $\sim$. Let α and β be orders and m be in ω.

(1) α is of *type* $\omega, \omega^*, \omega^* + \omega$, or $\omega + \omega^*$ respectively iff it is ω-equivalent to an order $\gamma = (\tau_1 + \cdots + \tau_n) \cdot \delta$ where the τ_i's are $\sim$-segments and $\delta \equiv_\omega \omega, \omega^*, \omega^* + \omega$, or $\omega + \omega^*$ respectively. The τ_i's, unique up to ω-equivalence, are called *components* of α, n, if it is the smallest possible, is the *period* of α, and $\tau_1 + \cdots + \tau_n$, unique up to ω-equivalence and possibly a cyclic permutation, is

the *cycle* of α. We shall also regard orders ω-equivalent to $(\tau_1 + \cdots + \tau_n)(\omega + \omega^*) + \tau_1 + \cdots + \tau_i$, $i < n$ as being of type $\omega + \omega^*$.

(2) α is *periodic* if it has type ω, ω^*, $\omega^* + \omega$, or $\omega + \omega^*$. α is of *integer type* if it has type $\omega^* + \omega$. α is of *unit type* if $\sim_\alpha$ has exactly one segment, in which case α is a *component* of itself.

(3) α is of *rational type* iff it is ω-equivalent to an order $\gamma = \eta(\tau_1, \cdots, \tau_n)$ where the τ_i's are $\sim_\gamma$-segments and η is the shuffle product operation. The τ_i's, unique up to ω-equivalence, are called *components* of α.

(4) α and β are *comparable* iff one is of rational type and the other is a component of the first.

(5) α is *relatively m-isolated* iff (a) α is of integer type and no elementarily distinct order of integer or unit type is m-equivalent to α or (b) α is of rational type and no elementarily distinct order which is of unit type and not comparable to α or which is of rational type is m-equivalent to α.

For any local splitting assignment $\sim$ and any order α periodic with respect to $\sim$: Let $\sim^\alpha$ be the splitting assignment defined by $x \sim^\alpha y$ iff $x \sim y$ or for some $\sim$-closed segment γ, x and y are in γ but not in the first or last n $\sim$-segments (called buffer zones) of γ and γ is ω-equivalent to an order of the form $\tau_i + \cdots + \tau_n + (\tau_1 + \cdots + \tau_n) \cdot \delta + \tau_1 + \cdots + \tau_j$ where the τ_k's are $\sim$-segments, $\tau_1 + \cdots + \tau_n$ is a cycle of α, $1 \leqq i, j \leqq n$, and either $\delta \in \omega$ or $\delta = \omega + \omega^*$. For example, if $\alpha = (\sigma + \tau + \sigma) \cdot \omega$ where σ and τ are $\sim$-segments then the segments of $\sim^\alpha$ in

$$(\sigma) + (\tau) + (\sigma) + (\sigma + \tau + \sigma) + (\sigma) + (\tau) + (\sigma) + (\tau) + (\sigma) + (\sigma + \tau + \sigma) + (\sigma) + (\tau) + (\sigma)$$

are as indicated by the parentheses. Without the buffer zones, transitivity would fail. If x and y are in γ_1 and y and z are in γ_2 where γ_1 and γ_2 satisfy the conditions on γ, then γ_1, γ_2, and their buffer zones overlap for at least one cycle of n $\sim$-segments and hence $\gamma_1 \cup \gamma_2$ satisfies the conditions on γ and hence $x \sim^\alpha z$. Note that all proper segments (segments which are not $\sim$-segments) of $\sim^\alpha$ are periodic with respect to $\sim$. Because of the buffer zones $\sim^\alpha$ will not be local.

If α is of rational type with respect to $\sim$, let $\sim^\alpha$ be defined by $x \sim^\alpha y$ iff $x \sim y$ or x and y are in a $\sim$-closed segment ω-equivalent to α. For periodic or rational type α, $\sim^\alpha$ is definable if $\sim$ is.

LEMMA 26. *For any definable local splitting assignment $\sim$, any order α, and any m: If α is of integer or rational type with respect to $\sim$ but is not relatively* *m-isolated, then α is m-equivalent to an order of rank ω with respect to a definable local splitting assignment extending $\sim$.*

PROOF. Let $\sim, \alpha$, and m be as in the hypothesis. Let β be an order m-equivalent but not ω-equivalent to α such that β has (a) integer or unit type (with respect to $\sim$) if α has integer type and (b) rational type or incomparable unit type if α has rational type. Pick $k \geqq \max(m, \text{quant}(\varphi)+2, \text{quant}(\sigma))$ for some φ defining $\sim$ and some σ distinguishing α and β. By Shelah's theorem we may replace the nonisolated $\sim$-segments of α by k-equivalent isolated orders. It suffices to prove the Lemma for the new α which is m-equivalent to the original and, by Lemma 12, still has the properties listed above. Moreover α now has rank 0 as does any integer or rational type with rank 0 components with respect to a definable splitting.

Since α is of integer or rational type and $\alpha \equiv_m \beta$, we have

$$\begin{aligned} \alpha &\equiv_m \alpha((2\cdot\eta + 2\cdot(\omega+\omega^*))\cdot\omega) \\ &= ((\alpha+\alpha)\cdot\eta + (\alpha+\alpha)(\omega+\omega^*))\cdot\omega \\ &\equiv_m ((\alpha+\beta)\cdot\eta + (\alpha+\beta)(\omega+\omega^*))\cdot\omega \\ &=_{\text{def}} \gamma. \end{aligned}$$

Our desired order of rank ω is γ. Since $\sim$ is local and since α has no first or last element mod $\sim$, $\sim_\gamma$ and $\sim_\alpha$ coincide on the α segments of γ. Hence the β segments of γ are unions of equivalence classes and hence, by localness, $\sim_\gamma$ and $\sim_\beta$ coincide on the β segments also.

We shall define two extensions $\sim'$ and $\sim^*$ of $\sim$ such that the segments of $\sim'_\gamma$ are the summands α and β and the segments of $\sim^*_\gamma$ are the summands $(\alpha+\beta)\cdot\eta$ and $(\alpha+\beta)(\omega+\omega^*)$. Since $(\alpha+\beta)\cdot\eta$ has rank 0 and $(\alpha+\beta)\cdot(\omega+\omega^*)$ has rank 1, γ has rank ω with respect to $\sim^*$. Hence it suffices to find such a $\sim^*$ which is definable and local.

Let σ_α be a sentence which isolates α. Let $\sim'$ be the splitting assignment defined by $x \sim' y$ iff $x \sim y$ or for some segment γ of $\sim^\alpha$ x and y are in γ and $\gamma = \sigma_\alpha$ or no point equal to or between x and y is in any such segment γ. Easily $\sim'$ is definable and the segments of $\sim'_\gamma$ are the summands α and β and it is not hard to verify that $\sim'$ satisfies the conditions of the definition of localness.

Let $\sim^*$ be the splitting assignment defined by $x \sim^* y$ iff $x \sim' y$ or x and y are in a segment which is discrete and has at least three elements mod $\sim'$ or x and y are in a segment which has order type $2\cdot\eta$ mod $\sim'$. Easily $\sim^*$ is definable and local and the segments of $\sim_\gamma^*$ are the summands $(\alpha+\beta)\cdot\eta$ and $(\alpha+\beta)\cdot(\omega+\omega^*)$.

Lemma 27. *For any definable local splitting assignment* $\sim$, *any order* α, *and any* m: *If* $\alpha/\sim\ \cong\omega$ *or* ω^* *and all integer-type-with-respect-to-*$\sim$ *segments of orders* m*-equivalent to* α *are relatively* m*-isolated, then* α *is eventually periodic with respect to* $\sim$.

Proof. Let $\sim$, α, and m be as in the hypothesis and suppose $\alpha/\sim\ \cong\omega$. Pick $k\geqq\max(m,\operatorname{quant}(\varphi)+2)$ for some φ defining $\sim$. By a Galvin-type argument (see Läuchli and Leonard [6] or Amit and Shelah [1]) $\alpha=\alpha_0+\Sigma_{i=1}^{\omega}\alpha_i$ where the α_i are finite unions of $\sim$ -segments and $\Sigma_{i=j}^{l}\alpha_i\equiv_k\Sigma_{i=p}^{q}\alpha_i$ for $1\leqq j\leqq l$ and $1\leqq p\leqq q$. Since $\alpha\equiv_m\alpha_0+\alpha_1\omega\equiv_\omega\alpha_0+\alpha_1\omega+\alpha_1(\omega^*+\omega)$, $\alpha_1(\omega^*+\omega)$ is an integer-type segment of an order m-equivalent to α. By hypothesis $\alpha_1(\omega^*+\omega)$ must be relatively m-isolated. Hence α_1 must be isolated.

Let σ_{α_1} be a sentence which isolates α_1, let σ be a sentence which says "there is a cofinal segment of type ω with respect to $\sim$ with a cycle which satisfies σ_{α_1}", and pick $n\geqq(\operatorname{quant}(\sigma),\operatorname{quant}(\varphi)+2$. Repeating the above argument we get $\alpha=\beta_0+\Sigma_{i=1}^{\omega}\beta_i$ where the β_i are finite unions of the α_i's and, as a special case, $\beta_i\equiv_n\beta_j$ for $i,j\geqq1$. Clearly $\alpha\equiv_n\beta_0+\beta_1\omega$. Also $\alpha_1\equiv_m\beta_1$ and hence $\alpha_1(\omega^*+\omega)\equiv_m\beta_1(\omega^*+\omega)$.

Since $\beta_1(\omega^*+\omega)$ has integer type, since $\alpha_1(\omega^*+\omega)$ is relatively m-isolated, and since $\alpha_1(\omega^*+\omega)\equiv_m\beta_1(\omega^*+\omega)$, we have $\alpha_1(\omega^*+\omega)\equiv_\omega\beta_1(\omega^*+\omega)$. So

$$\alpha\equiv_n\beta_0+\beta_1\omega\equiv_\omega\beta_0+\beta_1\omega+\beta_1(\omega^*+\omega)\equiv_\omega\beta_0+\beta_1\omega+\alpha_1(\omega^*+\omega)\models\sigma.$$

Since σ is an n-sentence, $\alpha\models\sigma$ and hence α is eventually periodic.

Lemma 28. *For any definable local splitting assignment* $\sim$, *any order* α, *and any* m: *If* $\alpha/\sim$ *is dense and all rational-type-with-respect-to-*$\sim$ *orders* m*-equivalent to segments of* α *are relatively* m*-isolated, then a segment of* α *has rational type with respect to* $\sim$.

Proof. Let $\sim$, α, and m be as in the hypothesis. Pick $k\geqq\max(m,\operatorname{quant}(\varphi)+2)$ for some φ defining $\sim$. Let β be a nonempty $\sim$-closed segment of α such that the number n of k-types of $\sim$-segments in β is minimum (compare Läuchli and Leonard [6]). Minimality implies that each of these k-types occurs densely in $\beta/\sim$. Let $\tau_1,\cdots,\tau_n$ be orders representing the n k-types. Suppose one of the k-types is not isolated. In particular, suppose $\tau_1'\equiv_k\tau$, but $\tau_1'\not\equiv_\omega\tau_1$. Then

$$\eta(\tau_1, \cdots, \tau_n) \equiv_k \eta(\tau'_1, \tau_2, \cdots, \tau_n) \equiv_k \beta$$

but $\eta(\tau_1, \cdots, \tau_n) \not\equiv_\omega \eta(\tau'_1, \tau_2, \cdots, \tau_n)$. This contradicts the assumption that all orders m-equivalent to segments of α are relatively m-isolated. Hence each of the n k-types is isolated and $\beta \equiv_\omega \eta(\tau_1, \cdots, \tau)$ is of rational type with respect to $\sim$.

LEMMA 29. *For any definable local splitting* $\sim$ *and any* m: *There are only finitely many elementarily distinct orders which are periodic (and hence infinite) with respect to* $\sim$ *and for which all integer-type-with-respect-to-*$\sim$ *segments of* m*-equivalent orders are relatively* m*-isolated.*

PROOF. Suppose $\sim$ and m are as in the hypothesis and suppose α is periodic with respect to $\sim$ and all integer segments of m-equivalent orders are relatively m-isolated. Then $\alpha \equiv_\omega (\tau_1 + \cdots + \tau_n) \cdot \delta$ where $\tau_1 + \cdots + \tau_n$ is a cycle and $\delta = \omega$, ω^*, $\omega^* + \omega$, or $\omega + \omega^*$. In any case δ is ω-equivalent to an order with an $\omega^* + \omega$ segment and hence α is ω-equivalent to an order with a $(\tau_1 + \cdots + \tau_n)(\omega^* + \omega)$ segment. Since, by hypothesis $(\tau_1 + \cdots + \tau_n)(\omega^* + \omega)$ must be relatively m-isolated and since there are only finitely many m-types, there are only finitely many elementarily distinct cycles $\tau_1 + \cdots, \tau_n$. Hence there are only finitely many elementarily distinct possibilities for α.

LEMMA 30. *For any definable local splitting assignment* $\sim$, *any order* α, *and any* m: *If* (1) *α is discrete (has no limit points)* mod $\sim$, (2) *all segments of* $\sim_\alpha$ *are isolated,* (3) *if* β *is a* $\sim$*-closed segment of* α *and* $\beta/\sim \;\cong \omega$ *or* ω^* *then* β *is eventually periodic with respect to* $\sim$ *and* (4) *for any periodic-with-respect-to-*$\sim$ *order* β, *no segment of a definable splitting* $\sim^\beta$ *extending* $\sim_\alpha$ *is of type* ω^*, $\omega^* + \omega$, $\omega + \omega^*$, *or* ω *with respect to* $\sim$ *with the exceptions that the first* $\sim^\beta$*-segment may have type* ω^*, *the last may have type* ω *and a* $\sim^\beta$*-segment which is first and last may have type* $\omega^* + \omega$. *Then* α *is isolated.*

PROOF. Let $\sim$, α, and m be as in the hypothesis. If α is finite mod $\sim$ it is clearly isolated. Otherwise it is infinite mod $\sim$ and α has a $\sim$-closed segment β such that $\beta/\sim \;\cong \omega$ or ω^*. By (3), we may assume that β is periodic with respect to $\sim$. Hence $\sim^\beta$ has an infinite-mod-$\sim$ segment γ. Since it is infinite and periodic, γ must have type ω, $\omega^* + \omega$, ω^*, or $\omega + \omega^*$ with respect to $\sim$. The last is ruled out by (4). In the remaining cases γ is isolated. If it has type $\omega^* + \omega$ then, by (4), $\gamma = \alpha$ and hence α is isolated. Otherwise γ has type ω or ω^*; suppose it has ω^*. By (4), $\alpha = \gamma + \alpha'$ for some $\sim$-closed segment α'. If α' is finite mod $\sim$, it is isolated and hence so is α (γ and α' are segments of a definable splitting of

α). Otherwise α is infinite and by repeating the above argument we get a $\sim$-closed segment γ' of α' of type ω^*, $\omega^* + \omega$, or ω. The first two are ruled out by (4) since no segment of α' can be a first segment of α. Hence γ' has type ω, it is isolated, and, by (4), $\alpha' = \alpha'' + \gamma'$ for some $\sim$-closed segment α''. If α'' is finite mod $\sim$, it and α are isolated. If it were infinite mod $\sim$, the above argument would give us a $\sim$-closed segment γ'' of α'' of type ω^*, $\omega^* + \omega$, or ω. But all of these cases are ruled out by (4) since no segment of α'' can be a first or last segment of α.

The classification lemma: proof

We now introduce a splitting assignment due essentially to Amit and Shelah [1]. For any splitting assignment $\sim$, the $\sim$-closure of an interval is called a *$\sim$-interval.*

Definition 31. For any local splitting assignment $\sim$ and any m:

(a) Let $\sim[m]$ be the splitting assignment defined by $x \sim [m] y$ iff $x \sim y$ or x and y are in a $\sim$-closed segment β all of whose $\sim$-intervals are m-equivalent to orders which are finite mod $\sim$.

(b) Let $\sim(m)$ be the splitting assignment defined by $x \sim (m) y$ iff $x \sim y$ or x and y are in a $\sim$-closed segment β which is dense and without endpoints mod $\sim$ and for which there are m-types $\tau_1, \cdots, \tau_n$ such that every $\sim$-segment in β is of type τ_i for some i and the set of $\sim$-segments in β of type τ_i is dense in $\beta/\sim$ for each $i \leqq n$.

Lemma 32. *If $\sim$ is a definable local splitting assignment, so are $\sim[m]$ and $\sim(m)$ for any m.*

Proof. The definable part is due essentially to Amit and Shelah [1]; the local part is easy.

Lemma 33. *For any local splitting assignment $\sim$, any order α, and any m: If $\sim_\alpha$ has more than one segment, then either $\sim[m]_\alpha$ or $\sim(m)_\alpha$ is a proper extension of $\sim_\alpha$.*

Proof. A simple Galvin-type argument. Let $\sim$, α, and m be as in the hypothesis. If $\sim[m]_\alpha \neq \sim_\alpha$ we are done. Otherwise $\sim[m]_\alpha = \sim_\alpha$ and hence there are no two consecutive $\sim_\alpha$-segments. Let β be a nonempty $\sim$-closed segment of α such that the number of m-types of $\sim$-segments in β is minimum. Minimality implies that each of these m-types occurs densely. Hence all points of β are $\sim(m)_\alpha$-equivalent but not all are $\sim_\alpha$-equivalent.

LEMMA 34. *For any definable local splitting assignment* $\sim$, *any order* α, *and any* m: *If* α *is not* m*-equivalent to an order of rank* ω *and if all segments of* $\sim_\alpha$ *are isolated, then there is a definable finite splitting* $\sim^*$ *extending* $\sim_\alpha$ *such that if* β *is a segment of* $\sim^*$ *then* β *has rank* 0 *or rank* 1 *or for some* n *all segments of* $\sim[n]_\beta$ *and* $\sim(n)_\beta$ *are isolated.*

PROOF. Let $\sim$, α, and m be as in the hypothesis. Suppose γ is a periodic (hence infinite) segment of α. By the hypothesis on α and Lemma 24, no segment of an order m-equivalent to γ has rank ω. By Lemma 26, all integer-type segments of orders m-equivalent to γ are relatively $\sim$-isolated. By Lemma 29 there are only finitely many elementarily distinct such orders γ. Suppose they are $\gamma_1, \cdots, \gamma_k$. Pick $m' \geqq \max(m, \text{quant}(\varphi)+2, \text{quant}(\varphi_1)\cdots, \text{quant}(\varphi_k)+2)$ for some φ defining $\sim$ and some φ_i defining $\sim^{\gamma_i}$. Pick $M \geqq m'$ such that for all $M' \geqq M$, $M' \equiv_{m'} \omega + \omega^*$.

By Lemma 12 and the choice of M, for any $\gamma \in \{\gamma_1, \cdots, \gamma_k\}$ and any segment δ of $\sim^\gamma$ in α, if δ has first and last points mod $\sim$ and at least M cycles and if α' and δ' are obtained from α and δ by replacing these M cycles with $\omega + \omega^*$ cycles, then $\alpha' \equiv_m \alpha$ and δ' is a rank 1 segment of $\sim^\gamma$ in α' of type $\omega + \omega^*$ with respect to $\sim$.

Define $\sim^*$ by $x \sim^* y$ iff (a) for some $\gamma \in \{\gamma_1, \cdots, \gamma_k\}$ x and y are in a segment of $\sim^\gamma$ in α which has first and last points mod $\sim$ and at least $M+1$ cycles or (b) neither x nor y nor any point between them is in any such segment. Easily $\sim^*$ is reflexive, symmetric, definable and convex i.e., $x < z < y$ and $x \sim^* y$ implies $x \sim^* z \sim^* y$. To show that $\sim^*$ is transitive it suffices to show that for distinct γ, $\gamma' \in \{\gamma_1, \cdots, \gamma_k\}$ no segment of $\sim^\gamma$ satisfying the conditions of (a) intersects any such segment of $\sim^{\gamma'}$. Suppose δ and δ' are intersecting segments of $\sim^\gamma$ and $\sim^{\gamma'}$ respectively which satisfy the conditions of (a) and suppose the period of δ is less than or equal to the period of δ'. If δ and its buffer zones (see definition of $\sim^\gamma$) cover an entire cycle in the union of δ' and its buffer zones, then δ and δ' must have the same periodic structure and $\gamma \equiv_\omega \gamma'$. Hence δ is properly included in some cycle ε in the union of δ' and its buffer zones. By assumption, δ' has $M+1$ repetitions of some cycle of $\sim^{\gamma'}$, hence it has M repetitions of cycles of type ε. Replace these M cycles by $\omega + \omega^*$ cycles isomorphic to ε. Each of these cycles of type ε includes a segment of $\sim^\gamma$ isomorphic to δ. By the previous paragraph each of these segments isomorphic to δ may be replaced by an m'-equivalent rank 1 segment. The resulting order is m-equivalent to α and has $\omega + \omega^*$ rank 1 segments contradicting the hypothesis on α. Hence $\sim^*$ is transitive. Finally $\sim^*$ is finite. Each segment of $\sim^*$ is either of type (a), i.e., all of its points are equivalent by condition (a), or is of type (b). There are only finitely many segments $\sim^*$ of type (a) since each such segment can be replaced by an m'-equivalent rank 1 segment. Since any two type (b) segments of $\sim^*$ must be separated by a type (a) segment, there are only finitely many type (b) segments.

Each segment $\sim^*$ of type (a) has rank 0 or rank 1. Suppose β is a segment of $\sim^*$ of type (b). We show that for some n, all segments of $\sim [n]_\beta$ and $\sim (n)_\beta$ are isolated.

Pick n so that $n \geqq m'$, $n \geqq \mathrm{quant}(\theta)$ for some θ which says "$<$ is discrete mod $\sim$" and for any $\gamma \in \{\gamma_1, \cdots, \gamma_k\}$ $n \geqq$ quant(σ) where σ says "every segment of $\sim^\gamma$ has first and last points mod $\sim$".

First we show that all segments of $\sim [n]_\beta$ are isolated. Suppose δ is a segment of $\sim [n]_\beta$. Every segment of $\sim_\beta$ is a segment of $\sim_\alpha$ and hence is isolated. Thus if δ is a segment of $\sim_\beta$ we are done. Otherwise δ is a proper segment of $\sim [n]_\beta$. We claim δ satisfies the hypotheses (1), (2), (3), and (4) of Lemma 30. (1) follows from the definition of $\sim [n]$ and the fact that the n-sentence "$<$ is discrete mod $\sim$" is true in all orders which are finite mod $\sim$. (2) holds since segments of $\sim_\gamma$ are segments of $\sim_\beta$ and hence isolated. (3) holds since if ε were a $\sim$-closed segment of δ and $\varepsilon/\sim\ \cong \omega$ or ω^* then by Lemma 24 no segment of an order m-equivalent to δ has rank ω, by Lemma 26 all integer-type segments of orders m-equivalent to δ are relatively m-isolated, and so, by Lemma 27, ε is eventually periodic with respect to $\sim$. To show (4), suppose ε is a segment of $\sim^\gamma$ for some periodic-with-respect-to-$\sim$ order γ and that ε has type ω, $\omega^* + \omega$, ω^* or $\omega + \omega^*$ with respect to $\sim$. By Lemma 26 and 24, γ is relatively m-isolated and hence $\gamma \in \{\gamma_1, \cdots, \gamma_k\}$. If ε had type $\omega + \omega^*$, it would be a type (a) segment contradicting the assumption that β has type (b). Hence ε has type ω, $\omega^* + \omega$, or ω^*. Suppose it has type ω^* and suppose it is not an initial segment of δ. Let x be a point preceding ε and let y be a point in ε. Let $\overline{[x, y]}$ be the $\sim$-closure of $[x, y]$. By definition of $\sim [n]$, $\overline{[x, y]}$ is n-equivalent to an order which is finite mod $\sim$. Hence, by choice of n, $\overline{[x, y]} \models$ "every segment of $\sim^\gamma$ has a first and last point mod $\sim$". Hence ε cannot have type ω^*. The case for ε of type ω and $\omega^* + \omega$ is similar.

Finally we show that all segments of $\sim (n)_\beta$ are isolated. Suppose δ is a segment of $\sim (n)_\beta$ which is not a segment of $\sim_\beta$. Then δ is dense and without endpoints mod $\sim$ and for some orders $\tau_1, \cdots, \tau_p$, every $\sim$-segment of δ is n-equivalent to τ_i for some i and the set of $\sim$-segments n-equivalent to τ_i is dense in $\delta/\sim$ for each i. Suppose one of the τ_i's, say τ_1, is not n-isolated. Suppose $\tau_1 \equiv_n \tau_1'$ and $\tau_1 \not\equiv_\omega \tau_1'$. Then, as in Lemma 28, $\eta(\tau_1, \cdots, \tau_p) \equiv_n \eta(\tau_1', \tau_2, \cdots, \tau_p)$ but $\eta(\tau_1, \cdots, \tau_p) \not\equiv_\omega \eta(\tau_1', \cdots, \tau_p)$. Hence

$\eta(\tau_1, \cdots, \tau_p)$ is of rational type but is not relatively n-isolated and hence not relatively m-isolated. By Lemma 26 and 24, α is m-equivalent to an order of rank ω contradicting the hypothesis. Hence the τ_i's are n-isolated and hence δ is a shuffle product of the τ_i's and is isolated.

LEMMA 35 (Classification lemma). *For any order α and any $m \in \omega$: Either α is m-equivalent to an order of rank ω or α has finite rank.*

PROOF. Given m, suppose β is not m-classifiable, i.e., suppose β is not m-equivalent to any order of rank ω and β does not have finite rank. Let $\sim$ be any definable local splitting assignment such that $\sim_\beta$ has only isolated segments. By Lemma 34 there is a finite definable extension $\sim^*$ of $\sim_\beta$ such that if γ is a segment of $\sim^*_\beta$ then γ has rank 0 or rank 1 or for some n, $\sim[n]_\gamma$ and $\sim(n)_\gamma$ have only isolated segments. Since β does not have finite rank, one of the segments of $\sim^*$ must not have finite rank. Let β' be such a segment. By Lemma 24 and the hypothesis on β, β' is not m-equivalent to any order of rank ω. Hence β' is nonclassifiable. Since β' can not be of rank 0 or rank 1, there is an n, pick one, such that $\sim[n]_{\beta'}$ and $\sim(n)_{\beta'}$ have only isolated segments. Since β' does not have finite rank, $\sim_{\beta'}$ must have more than one segment and hence, by Lemma 33, either $\sim[n]_{\beta'}$ or $\sim(n)_{\beta'}$ is a proper extension of $\sim_{\beta'}$. Let $\sim'$ be $\sim[n]_{\beta'}$ if it is a proper extension, otherwise let it be $\sim(n)_{\beta'}$. By Lemma 32, $\sim'$ is a definable, local splitting assignment. We already have that $\sim'_{\beta'}$ has only isolated segments.

Now suppose for the sake of a contradiction that α_0 is not m-classifiable. Let $\sim_0$ be equality. Then $\sim_0$ is definable local splitting assignment which has only isolated segments in α_0. Given α_i and $\sim_i$, let α_{i+1} be α'_i and $\sim_{i+1}$ be $\sim'_i$ as in the previous paragraph. By induction, α_i is not m-classifiable, $\sim_i$ is a local definable splitting assignment, $\sim_i$ has only isolated segments in α_i, and $\sim_{i+1}$ properly extends $\sim_i$ on α_{i+1}.

Let I be the set of segments β of α_0 such that for some i, β is a segment of $\sim_{i+1}$ and is infinite and discrete mod $\sim_i$. Let R be the set of segments β of α_0 such that for some i, β is a segment of $\sim_{i+1}$ and is infinite and dense mod $\sim_i$. We shall say β is a proper segment of $\sim_{i+1}$ if β is a segment of $\sim_{i+1}$ but not of $\sim_i$. Since α_0 is not m-equivalent to any order of rank ω, Lemmas 24 and 26 imply that all integer or rational-type segments of orders m-equivalent to segments in I or R must be relatively m-isolated.

Given i suppose no segment of $\sim_{i+2}$ or $\sim_{i+1}$ is in I or R. Since every proper segment of $\sim(n)$ is infinite and dense mod $\sim$, $\sim_{i+2} = \sim_{i+1}[n]$ for some n and $\sim_{i+1} = \sim_i[n']$ for some n'. Since $\sim_{i+2}$ properly extends $\sim_{i+1}$ on α_{i+2}, there is a proper segment β of $\sim_{i+1}$ in α_{i+2}. Since β is not in I, β is finite mod $\sim_{i+1}$. For the same reason every $\sim_{i+1}$-segment of β is finite mod $\sim_i$. But then β is finite mod $\sim_i$ and hence a segment of $\sim_{i+1}$ and not a proper segment of $\sim_{i+2}$. Hence either I or R has proper segment from $\sim_{i+1}$ for infinitely many i.

Suppose for the sake of a contradiction that I has proper segments from $\sim_{i+1}$ for infinitely many i. Let P be the set of i such that I has a proper segment from $\sim_{i+1}$. For each i in P pick a segment β in I properly from $\sim_{i+1}$. Since β is infinite and discrete mod $\sim_i$ it must include a segment β' such that $\beta'/\sim_i \cong \omega$ or ω^*. Since all integer-type segments of orders m-equivalent to β' must be relatively m-isolated, β' is eventually periodic with respect to $\sim_i$ by Lemma 27. By a compactness argument, β' is ω-equivalent to an order which includes a segment β_i of integer type with respect to $\sim_i$. Since β_i is an integer segment of an order m-equivalent to β', β_i must be relatively m-isolated with respect to $\sim_i$. Since P is infinite there are infinitely many such β_i's and hence for some $i < j$ in P, $\beta_i \equiv_m \beta_j$. Since β_i has unit type with respect to $\sim_{i+1}$ and hence with respect to $\sim_j$ and since β_j has rational type with respect to $\sim_j$, $\beta_i \not\equiv_\omega \beta_j$. But then β_j is not relatively isolated with respect to $\sim_j$. Hence a contradiction.

Suppose for the sake of a contradiction that R has proper segments from $\sim_{i+1}$ for infinitely many i. Let P be the set of i such that R has a proper segment from $\sim_{i+1}$. For each $i \in P$ pick a segment β in R properly from $\sim_{i+1}$. Since all segments of orders m-equivalent to β must be relatively m-isolated, β includes a segment β_i of rational type with respect to $\sim_i$ by Lemma 28. Being a rational-type segment of β, β_i must be relatively m-isolated with respect to $\sim_i$. Since P is infinite there are infinitely many such β_i's and hence for some $i < j < k$ in P, $\gamma \equiv_m \delta \equiv_m \varepsilon$ where $\gamma = \beta_i$, $\delta = \beta_j$, and $\varepsilon = \beta_k$. Since $i < j < k$, γ has unit type with respect to $\sim_j$ and γ and δ have unit type with respect to $\sim_k$. Hence γ, δ, and ε are elementarily distinct. If any two of them are incomparable, they are not relatively m-isolated and we have a contradiction. Suppose they are all comparable. Hence $\varepsilon \equiv_\omega \eta(\gamma, \delta, \cdots)$ where $\cdots$ are the components of ε other than (there is at least one) γ and δ.

Let $\varepsilon' = \eta(\_, \delta, \cdots)$ where $\eta(\_, \delta, \cdots)$ is $\eta(\gamma, \delta, \cdots)$ with all the $\sim_k$-segments of type γ deleted. A consideration of the definition of the splitting $\sim(N)$ shows that ε' has rational type with respect to $\sim_n$ for some $n \leqq k$. We claim γ and ε' are incomparable. Since δ is a segment of ε', $j \leqq n$ and, since $i < j$, ε' is not a component of γ with respect to $\sim_i$. Showing that γ is not a component of ε' is not entirely trivial ($\eta(1) \cong \eta(1, \eta) \cong \eta(\eta)$ but only 1 is a component with respect to equality). Suppose γ were a component of ε' with respect to $\sim_n$ and let γ_0 be a particular $\sim_n$-segment of ε' of type γ. No component of ε' with respect to $\sim_n$ can contain a cut determined by an omitted

γ_i; if it did it would contain a segment isomorphic to ε' which is impossible. Hence $\varepsilon' \cong \varepsilon$ since ε has only an additional dense set of γ-type segments. Hence γ_0 would also be a γ-type segment of $\sim_n$ in ε contradicting the assumption that all such segments had been omitted. Hence γ and ε' are incomparable and, as a consequence, elementarily distinct. Since $\gamma \equiv_m \delta$, $\varepsilon' = \eta(\_, \delta, \cdots) \cong \eta(\delta, \delta, \cdots) \equiv_m \eta(\gamma, \delta, \cdots) = \varepsilon \equiv_m \gamma$. Hence ε' is not relatively m-isolated. This contradicts the fact that all rational-type segments of orders m-equivalent to segments in R are relatively m-isolated. Hence the Lemma.

Conclusion

Among the decidable theories whose Boolean sentence algebras have been characterized are: equality, equivalence relations [3], unary functions [9], well-orders [7], algebraically closed fields [11], real-closed fields [11], Boolean algebras [10], and abelian groups [7].

The undecidable axiomatizable theories where sentence algebras have been characterized tend to fall into two groups: (1) the essentially undecidable theories whose sentence algebras are the atomless Boolean algebra and (2) the theories whose sentence algebras are isomorphic to that of the theory of binary relations (characterized in Hanf [4]). Among the latter are the theories of lattices, semigroups and of any finite undecidable similarity type.

Open Problem. Characterize the Boolean sentence algebras of the theory of groups and the theory of distributive lattices.

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